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LETTER TO THE EDITOR

A Hopf bifurcation breaking rotation symmetry

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Abstract. We show that a double-degenerate Hopf problem, exhibiting covariance with respect to the rotation group SO(2), can admit a bifurcating periodic solution which breaks this symmetry.

It is a well known property of bifurcation phenomena in the presence of symmetry [1, 2] that the occurrence of a bifurcation usually corresponds to a breakdown of this symmetry; in fact, the branching solution exhibits, in general, a strictly lower symmetry than the original problem. The case when the symmetry is described by the group SO(2) is in some sense a very special case: there is in fact a close connection of this symmetry to Hopf-type bifurcations [1-3]. On the other hand, it is known that periodic branching solutions to standard two-dimensional SO(2)-covariant Hopf problems actually preserve this covariance [1-4]. The purpose of this letter is to show a mechanism by which a bifurcation problem exhibiting covariance under the rotation group SO(2), and in the presence of multiple critical imaginary eigenvalues, admits a bifurcating periodic solution which breaks this symmetry.

Consider a four-dimensional bifurcation problem of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \omega\frac{\mathrm{d}u}{\mathrm{d}\tau} = f(\lambda, u) \qquad u = u(t) \qquad u \in R^4, \lambda \in R \tag{1}$$

with the usual rescaling $t \to \tau = \omega t$ (in such a way that one has to look for 2π-periodic solutions in τ), and where $f: R \times R^4 \to R^4$ is assumed to be smooth (e.g. analytical, for simplicity), with $f(\lambda, 0) = 0$.

We assume now, explicitly, that (1) is a 'double-degenerate' Hopf problem, and that it is covariant under the rotation group SO(2). More precisely:

(i) there is a 'critical value' $\lambda = \lambda_0$ of the control parameter λ for which the linearised part of f (the prime denotes differentiation)

$$L(\lambda) \equiv f'_u(\lambda, 0)$$

possesses two imaginary eigenvalues $\pm i\omega_0$ with double (geometrical and algebraic) multiplicity, and

(ii) there is a reducible representation

$$T = T_1 \oplus T_2$$

of SO(2), where T_1 and T_2 are equivalent to the fundamental real orthogonal two-dimensional representation T_0 of SO(2), such that

$$f(\lambda, Tu) = Tf(\lambda, u). \tag{2}$$

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As a first step, starting from (i) and (ii), it is easy to see that it is possible to perform a linear change of coordinates u, in such a way that, with respect to the new coordinates, $L(\lambda_0)$ takes the form

$$L_0 \equiv L(\lambda_0) = \omega_0 \begin{pmatrix} J & 0 \\ 0 & \pm J \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

For simplicity, we shall use the same notation with respect to the new variables; from now on, we shall always refer to this new system of coordinates. In addition, the SO(2) covariance is preserved; precisely, the new $f(\lambda, u)$ still satisfies (2), where now

$$T = \begin{pmatrix} T_0 & 0 \\ 0 & T_0 \end{pmatrix}. \tag{4}$$

Note the two alternative possibilities (the '$\pm$' sign in (3)) for L_0: it is impossible, in fact, to further reduce L_0 into a unique form without destroying the SO(2) covariance.

Due to the multiplicity of the critical eigenvalues, in order to ensure the existence of a bifurcating solution (cf [5]), we need another 'weak' form of symmetry for the map $f(\lambda, u)$, i.e.

(iii) there is a linear operator $A(\neq I)$ posessing the eigenvalue $\alpha = 1$, such that

$$f(\lambda, Au) = Af(\lambda, u).$$

In particular, as a consequence of (i) and (3), one has

$$L_0 A = A L_0 \tag{5}$$

and the geometrical multiplicity of the eigenvalue $\alpha = 1$ of A is necessarily equal to two. The following result then holds.

Lemma. Let $V_1 = R^2$ be the subspace spanned by the eigenvectors of A with eigenvalue $\alpha = 1$. Then

$$f_1 \equiv f|_{V_1}: R \times V_1 \rightarrow V_1$$

and so (1) admits a restriction to V_1.

The proof follows easily, taking $v \in V_1$, from

$$Af(\lambda, v) = f(\lambda, Av) = f(\lambda, v) \in V_1.$$

If now L_0 in (3) has the sign '+', then the reduced problem to V_1 still exhibits the SO(2) covariance: indeed, as a consequence of (ii) and (3)-(5), $f_1(\lambda, v)$ is covariant with respect to the representation T_0 acting on V_1. The same is *not* true in general if the sign in (3) is '$-$'. Consider, for instance, the case in which A has the special form

$$A = \begin{pmatrix} 0 & S^{-1} \\ S & 0 \end{pmatrix} \tag{6}$$

where S is a 2×2 real non-singular matrix. Then, writing

$$u = (u_1, u_2, u_3, u_4) \equiv (x_1, x_2, y_1, y_2)$$

$$f = (f_1, f_2, f_3, f_4) \equiv (X_1, X_2, Y_1, Y_2)$$

assumption (i) becomes, with $X \equiv (X_1, X_2)$, etc,

$$Y(\lambda; x, y) = SX(\lambda; S^{-1}y, Sx) \qquad x \in R^2, y \in R^2$$

and V_1 is spanned by the vectors $v \equiv (x, Sx)$. Once again, (cf [3, 5]), one may remark upon the reduction in the dimensionality of the problem, operated by the Z_2 symmetry generated by the operator A (in the form (6), $A^2 = I$). The reduced problem can, in fact, be written in the form

$$\mathrm{d}x/\mathrm{d}t = X(\lambda; x, Sx) \qquad x \in R^2. \tag{7}$$

If the sign in (3) is now '−', one has from (5) that $SJ = -JS$, and then S has the form

$$S = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \rho\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_\theta \tag{8}$$

for some $\rho > 0$ and $R_\theta \in T_0 \simeq \mathrm{SO}(2)$, and in this case the reduced equation (7) is in general *not* symmetric under SO(2).

We now summarise the above results, considering for simplicity only the case (6), and observing also that, if in (3) the sign is '+', then $SJ = JS$ and therefore

$$S = aI + bJ = \rho R_\theta. \tag{9}$$

Theorem. Given (1), let (i), (ii) and (iii) be verified, with A given by (6). Let the linearised part, put in the form (3), have the sign '+'. The problem can then be reduced to a standard two-dimensional Hopf problem which is SO(2) covariant. Assuming standard transversality conditions, there is a bifurcating solution of the form

$$x(t) = r\begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix} \qquad y(t) = \rho r\begin{pmatrix} \cos(\omega t + \theta) \\ \sin(\omega t + \theta) \end{pmatrix} = Sx(t)$$

$$\lambda = \lambda(r) \qquad \text{with } \lambda(r) \to \lambda_0 \qquad \text{when } r \to 0$$

$$\omega = \omega(r) \qquad \text{with } \omega(r) \to \omega_0 \qquad \text{when } r \to 0$$

where r is a real parameter defined in a neighbourhood of the origin, and for some fixed ρ, θ. This solution preserves the SO(2) covariance; indeed, $T_0 x(t)$ and $T_0 y(t)$ also solve the same problem, and in fact only the 'fundamental frequency' ω appears in the solution. If instead the linear part (3) has the sign '−', one obtains a two-dimensional reduced equation which is in general not SO(2) covariant, and the bifurcating solution has the form

$$x(t) = r\begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix} + \text{HOT} \qquad y(t) = \rho r\begin{pmatrix} \cos(\omega t + \theta) \\ -\sin(\omega t + \theta) \end{pmatrix} + \text{HOT}$$

where now HOT are higher-order terms (in the parameter r) containing higher-order harmonics.

A very simple explicit example in which all the above assumptions are verified, a bifurcating solution exists, and which actually breaks the initial SO(2) symmetry or not, depending on the sign in the second equation, is the following:

$$\mathrm{d}x/\mathrm{d}t = Jx + \lambda x + y|y|^2$$

$$\mathrm{d}y/\mathrm{d}t = \pm Jy + \lambda y + x|x|^2$$

where $|\cdot|$ is the standard R^2 norm.

It is not difficult, of course, to extend the above method and results to the case of larger multiplicities M (i.e. $M > 2$) of the critical eigenvalues of L_0, in $2M$-dimensional problems.

As a final remark, let us emphasise the crucial role played in our discussion by the operator A (or S, if in the form (6)). In fact, one can see that the SO(2) symmetry is broken or not depending on whether S has the form (8) or (9). In the above discussion, we referred to the classical Hopf procedure just for definiteness: we therefore had to deal with the linear term (3) and the double possibility for the sign in it, as explained. But it is clear that, being essentially based on group-theoretical ideas, our arguments—and in particular the occurrence of the breaking of the SO(2) covariance—hold equally even if different specific hypotheses are assumed in order to have bifurcations. For instance, the reduction of the problem to a two-dimensional form (7) can allow, depending on the form of the function X, the use of the classical Poincaré-Bendixson results in order to ensure the presence of bifurcating limit cycles [6, 7]. Similarly, stability exchange arguments can equally well be used within our scheme: we mention here, as a typical result, the fact that, if at the critical point λ_0 the trivial zero solution is asymptotically stable, and for $\lambda > \lambda_0$ it becomes completely unstable, then a stable bifurcating solution appears [8, 9]. In our case, these stability properties of the solution $x = 0$ can be checked, by means of Lyapunov function techniques (see, e.g., [10]) directly on the function X in (7).

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